# Energy Contents of Some Non-Vacuum Spacetimes in Teleparallel Gravity

M. Sharif*and Sumaira Taj†
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

**Abstract**

This paper elaborates the problem of energy-momentum in the framework of teleparallel equivalent of General Relativity. For this purpose, we consider energy-momentum prescription derived from the integral form of the constraint equations developed in the Hamiltonian formulation of the teleparallel equivalent of General Relativity. We use this technique to investigate energy-momentum of stationary axisymmetric Einstein-Maxwell solutions and cosmic string spacetimes. The angular momentum, gravitational and matter energy-momentum fluxes of these spacetimes are also evaluated. It is concluded that the results of teleparallel theory are relatively analogous to the results of General Relativity.

**Keywords**: Teleparallel Gravity; Energy-Momentum.

## 1 Introduction

The localization problem of energy-momentum emerged along with the field equations and still inconclusive. The conservation laws of energy and momentum for gravitation are major causes of the problem. The localization is

*msharif@math.pu.edu.pk
†sumairataj@ymail.com

not debatable for flat spacetime. However, for curved spacetime, the energy-momentum tensor of matter plus gravitation fields no longer satisfies the conservation law. Various energy-momentum prescriptions have been proposed to overcome this problem. The first of such contributions came from Einstein [1], who gave an expression for energy-momentum due to matter, non-gravitational and gravitational fields. After this, Møller [2], Landau-Lifshitz [3], Papapetrou [4], Bergmann [5], Tolman [6], Weinberg [7] and Komar [8] proposed their prescriptions. The key issue with energy-momentum prescriptions is their coordinate dependence and non-uniqueness. All these prescriptions, except Møller and Komar, give acceptable results only when one uses Cartesian coordinates.

Virbhadra and his collaborators [9]-[10] considered several spacetimes and verified that different prescriptions could give the same results for a given spacetime. Radinschi [11] evaluated energy distribution of Bianchi type I universe and found the same results as those of Banerjee and Sen [12] and Xulu [13]. In a recent paper, Abbassi et al. [14] have proved that various prescriptions provide the same results in static and non-static cosmic string spacetimes. Sharif [15] have shown that there exist several spacetimes that do not yield the same results for different prescriptions.

Some authors [16]-[17] argued that the localization problem of energy-momentum becomes more transparent in the framework of teleparallel equivalent of General Relativity (GR). Møller [18] was the first who observed that the tetrad description of the gravitational field could lead to a better expression for the gravitational energy-momentum than does GR. Mikhail et al. [19] derived the energy-momentum complex in Møller's tetrad theory. Loi So and Vargas [20] investigated that the quasi-local energy-momentum of Bianchi types I and II universes vanishes everywhere in the context of teleparallel gravity which coincides with the result in GR. In recent papers, Sharif and Amir [21]-[22] concluded that in teleparallel theory, the results of energy-momentum prescriptions did not coincide with GR for a given spacetime.

Andrade et al. [23] considered the localization of energy in Lagrangian framework of teleparallel equivalent GR (TEGR). Maluf et al. [24] derived the expression for the gravitational energy, momentum and angular momentum from the Hamiltonian formulation of TEGR [25]. Maluf computed gravitational energy of the de Sitter spacetime [26] and gravitational pressure for the Schwarzschild spacetime [27] in the framework of TEGR. da Rocha-Neto and Castello-Branco [28] extended this procedure to evaluate gravitational

energy of the Kerr and Kerr anti-de Sitter spacetimes. In a recent paper, Maluf and Ulhoa [29] and the authors [30] discussed gravitational energy-momentum of gravitational waves.

In this paper, we elaborate the above procedure by evaluating energy and its relevant quantities for stationary axisymmetric solutions of the Einstein-Maxwell field equations and cosmic string spacetimes. The paper has been organized in the following fashion: Section 2 presents some elementary notions of TEGR and energy-momentum expressions. In section 3, we calculate energy and its contents for stationary axisymmetric solutions. Section 4 is devoted for the two special cases of stationary axisymmetric solutions of the Einstein-Maxwell field equations. Energy and its contents for cosmic string spacetimes are discussed in section 5. In section 6, we summarize and discuss the results.

Here, the spacetime indices $(\mu, \nu, \rho, ...)$ and tangent space indices $(a, b, c, ...)$ run from 0 to 3. Time and space indices are denoted according to $\mu = 0, i$, $a = (0), (i)$.

## 2 Hamiltonian Approach: Energy-Momentum in Teleparallel Theory

The torsion tensor in terms of tetrad field is defined as

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu, \tag{2.1}$$

which is related to the Weitzenböck connection [31]

$$\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda \partial_\nu e^a{}_\mu. \tag{2.2}$$

The Lagrangian density for the gravitational field in TEGR is [25]

$$L = -\kappa e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a) - L_M \equiv -\kappa e \Sigma^{abc}T_{abc} - L_M, \tag{2.3}$$

where $\kappa = 1/16\pi$ and $e = det(e^a{}_\mu)$. The tensor $\Sigma^{abc}$ is defined as

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c). \tag{2.4}$$

The corresponding field equations become

$$e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = \frac{1}{4\kappa}eT_{a\mu}, \tag{2.5}$$

where

$$\frac{\delta L_M}{\delta e^{a\mu}} = eT_{a\mu}.$$

The total Hamiltonian density is [32]

$$H(e_{ai}, \Pi_{ai}) = e_{a0}C^a + \alpha_{ik}\Gamma^{ik} + \beta_k\Gamma^k + \partial_k(e_{a0}\Pi^{ak}), \tag{2.6}$$

where $C^a$, $\Gamma^{ik}$ and $\Gamma^k$ are primary constraints, $\alpha_{ik}$ and $\beta_k$ are the Lagrangian multipliers. In the constraint $C^a$, the first term is given by a total divergence

$$C^a = -\partial_i\Pi^{ai} + H^a, \quad \text{where } \Pi^{\rm ai} = -4\kappa \mathrm{e}\Sigma^{\rm a0i}, \tag{2.7}$$

$C^a$ is the momentum canonically conjugated to $e_{ai}$. The term $-\partial_i\Pi^{ai}$ is the *energy-momentum density* [24]. The angular momentum can be defined from the constraint $\Gamma^{ik} = 0$ giving the *angular momentum density.*

$$2\Pi^{[ik]} = 2\kappa e[-g^{im}g^{kj}T^0{}_{mj} + (g^{im}g^{0k} - g^{km}g^{0i})T^j{}_{mj}]. \tag{2.8}$$

The field equations (2.5) can also be written as

$$-\partial_0(\partial_j\Pi^{aj}) = -\kappa\partial_j[ee^{a\mu}(4\Sigma^{bcj}T_{bc\mu} - \delta^j_\mu\Sigma^{bcd}T_{bcd})] - \partial_j(ee^a{}_\mu T^{j\mu}). \tag{2.9}$$

Integration yields

$$\frac{d}{dt}[-\int_V d^3x\partial_j\Pi^{aj}] = -\Phi^a_g - \Phi^a_m, \tag{2.10}$$

where

$$\Phi^a_g = \int_S dS_j\phi^{aj}, \quad \Phi^a_m = \int_S dS_j(ee^a{}_\mu T^{j\mu}) \tag{2.11}$$

are the $a$ components of the *gravitational and matter energy-momentum flux* respectively [33]. $S$ represents the spatial boundary of the volume $V$. The quantity

$$\phi^{aj} = \kappa ee^{a\mu}(4\Sigma^{bcj}T_{bc\mu} - \delta^j_\mu\Sigma^{bcd}T_{bcd}) \tag{2.12}$$

is the $a$ component of the gravitational energy-momentum flux density in $j$ direction. In terms of the gravitational energy-momentum, Eq.(2.10) takes the form

$$\frac{dP^a}{dt} = -\Phi^a_g - \Phi^a_m. \tag{2.13}$$

Since $P^a = (E, \mathbf{P})$, thus *the loss of gravitational energy* is defined by

$$\frac{dE}{dt} = -\Phi^{(0)}_g - \Phi^{(0)}_m. \tag{2.14}$$

## 3 Stationary Axisymmetric Einstein-Maxwell Solutions

It has been the subject of interest over the past decades to study the behavior of the electromagnetic field in the strong gravitational field. Electromagnetic fields, in particularly magnetic fields play a significant role in astrophysics (neutron stars, white dwarfs and galaxy formation). A comprehensive relativistic insight of such situations require studying the Einstein-Maxwell equations. Tupper [34] gave a generalized metric, which contains five classes of non-null electromagnetic field plus perfect fluid solutions. Its metric has a symmetry not inherited by the electromagnetic field. Out of these five classes, two contain electrovac solutions as special cases and the remaining three classes must contain fluid. The general form of the line element describing the stationary axisymmetric solutions of the Einstein-Maxwell field equations is

$$ds^2 = -dt^2 + e^{2K}d\rho^2 + (F^2 - B^2)d\phi^2 + e^{2K}dz^2 - 2Bdtd\phi, \tag{3.1}$$

where the functions $B = B(\rho, z)$, $K = K(\rho, z)$ and $F = F(\rho)$ satisfy the following conditions

$$\begin{aligned} \dot{B} &= FW', \quad B' = -F\dot{W}, \quad \dot{K} = -\frac{1}{4}aF(\dot{W}^2 - W'^2), \\ K' &= -\frac{1}{2}aF\dot{W}W', \quad \ddot{W} + \dot{F}F^{-1}\dot{W} + W'' = 0. \end{aligned} \tag{3.2}$$

Here dot and prime denote differentiation with respect to $\rho$ and $z$ respectively, $B$ and $F$ have the dimension of length and $K$ is dimensionless, $W$ is an arbitrary function of $\rho$ and $z$, and $a$ is a constant. The tetrad field corresponding to the line element (3.1) is

$$e^a{}_\mu(\rho, \phi, z) = \begin{pmatrix} 1 & 0 & B & 0 \\ 0 & e^K\cos\phi & -F\sin\phi & 0 \\ 0 & e^K\sin\phi & F\cos\phi & 0 \\ 0 & 0 & 0 & e^K \end{pmatrix}, \tag{3.3}$$

and its determinant is $e = Fe^{2K}$. The non-vanishing components of the torsion tensor are

$$\begin{aligned} T_{(0)12} &= -\dot{B}, \quad T_{(0)23} = B', \quad T_{(1)12} = (e^K - \dot{F})\sin\phi, \\ T_{(1)13} &= -K'e^K\cos\phi, \quad T_{(2)12} = (\dot{F} - e^K)\cos\phi, \\ T_{(2)13} &= -K'e^K\sin\phi, \quad T_{(3)13} = \dot{K}e^K. \end{aligned} \tag{3.4}$$

Consequently, the components of the tensor $T_{\lambda\mu\nu} = e^a{}_\lambda T_{a\mu\nu}$ become

$$\begin{aligned} T_{012} &= -\dot{B}, \quad T_{023} = B', \quad T_{113} = -K' e^{2K}, \\ T_{212} &= -B\dot{B} + F(\dot{F} - e^K), \quad T_{223} = BB', \quad T_{313} = \dot{K} e^{2K}. \end{aligned} \tag{3.5}$$

## 3.1 Energy, Momentum and Angular Momentum

The non-zero components of energy-momentum density associated with the metric (3.1) are found using Eqs.(2.7) and (2.4)

$$\begin{aligned} -\partial_i \Pi^{(0)i} &= 2\kappa \left( \dot{K} e^K - \ddot{F} - F\ddot{K} - \dot{F}\dot{K} + \frac{\dot{B}^2}{2F} + \frac{B\ddot{B}}{2F} - \frac{B\dot{B}\dot{F}}{2F^2} + FK'' \right. \\ &\quad \left. + \frac{B'^2}{2F} + \frac{BB''}{2F} \right), \\ -\partial_i \Pi^{(1)i} &= 2\kappa \sin\phi \left( \frac{\ddot{B}}{2} + \dot{B}\dot{K} + B\ddot{K} - \frac{\dot{B} e^K}{2F} + \frac{B''}{2} - B'K' - BK'' \right), \\ -\partial_i \Pi^{(2)i} &= 2\kappa \cos\phi \left( -\frac{\ddot{B}}{2} - \dot{B}\dot{K} - B\ddot{K} + \frac{\dot{B} e^K}{2F} - \frac{B''}{2} + B'K' + BK'' \right). \end{aligned} \tag{3.6}$$

It is interesting to note that the energy density $-\partial_i \Pi^{(0)i}$ turn out equivalent to the energy density obtained in teleparallel theory using the Møller prescription [35] and can be re-written as

$$-\partial_i \Pi^{(0)i} = (-\partial_i \Pi^{(0)i})_{GR} + 2\kappa(\dot{K} e^K - \ddot{F} - F\ddot{K} - \dot{F}\dot{K} + FK'').$$

Integration of Eq.(3.6) yields energy

$$\begin{aligned} P^{(0)} &= 2\kappa \int_V d^3x \left( \dot{K} e^K - \ddot{F} - F\ddot{K} - \dot{F}\dot{K} + \frac{\dot{B}^2}{2F} + \frac{B\ddot{B}}{2F} - \frac{B\dot{B}\dot{F}}{2F^2} \right. \\ &\quad \left. + FK'' + \frac{B'^2}{2F} + \frac{BB''}{2F} \right) \end{aligned} \tag{3.7}$$

and momentum vanishes.

The components of angular momentum density $2\Pi^{[ik]}$ are

$$\begin{aligned} 2\Pi^{[11]} &= 0 = 2\Pi^{[13]}, \quad 2\Pi^{[22]} = 0 = 2\Pi^{[33]}, \\ 2\Pi^{[12]} &= -\frac{2\kappa}{F}(\dot{B} + B\dot{K}) = -2\Pi^{[21]}, \\ 2\Pi^{[23]} &= \frac{2\kappa}{F}(B' - BK') = -2\Pi^{[32]}. \end{aligned} \tag{3.8}$$

Consequently, the components of angular momentum become

$$\begin{aligned}
M^{11} &= \text{constant} = M^{13}, \quad M^{22} = \text{constant} = M^{33}, \\
M^{12} &= -2\kappa \int_V d^3x \frac{1}{F}(\dot{B} + B\dot{K}) = -M^{21}, \\
M^{23} &= 2\kappa \int_V d^3x \frac{1}{F}(B' - BK') = -M^{32}.
\end{aligned} \tag{3.9}$$

## 3.2 Energy-Momentum Flux

Here we discuss gravitational as well as matter energy-momentum fluxes due to the non-vacuum spacetimes. The components of gravitational energy flux density $\phi^{(0)j}$ are

$$\phi^{(0)1} = 0 = \phi^{(0)2} = \phi^{(0)3}. \tag{3.10}$$

Substituting these values in Eq.(2.11) for $a = (0)$, the gravitational energy flux becomes

$$\Phi_g^{(0)} = \text{constant}. \tag{3.11}$$

The momentum flux $\Phi_g^{(1)}$ for $a = (1)$ gives

$$\Phi_g^{(1)} = \int_S dS_j \phi^{(1)j}, \tag{3.12}$$

where

$$\begin{aligned}
\phi^{(1)1} &= 2\kappa \cos\phi e^{-K}\left(-\frac{\dot{B}^2}{4F} + FK'^2 + \dot{K}e^K - \dot{F}\dot{K} + \frac{B'^2}{4F}\right), \\
\phi^{(1)2} &= 2\kappa \sin\phi \left(\frac{\dot{B}^2}{4F^2} + \frac{B'^2}{4F^2} + K'^2\right), \\
\phi^{(1)3} &= 2\kappa \cos\phi e^{-K}\left(-\frac{\dot{B}B'}{2F} - K'e^K + \dot{F}K'\right).
\end{aligned} \tag{3.13}$$

Replacing these values in Eq.(3.12), it follows that

$$\Phi_g^{(1)} = 2\kappa \sin\phi \int_S dS_2 \left(\frac{\dot{B}^2}{4F^2} + \frac{B'^2}{4F^2} + K'^2\right). \tag{3.14}$$

Similarly, we can find $\Phi_g^{(2)}$ using the components of momentum flux density $\phi^{(2)j}$

$$\begin{aligned}
\phi^{(2)1} &= 2\kappa \sin\phi e^{-K}\left(-\frac{\dot{B}^2}{4F} + FK'^2 + \dot{K}e^K - \dot{F}\dot{K} + \frac{B'^2}{4F}\right), \\
\phi^{(2)2} &= -2\kappa \cos\phi \left(\frac{\dot{B}^2}{4F^2} + \frac{B'^2}{4F^2} + K'^2\right), \\
\phi^{(2)3} &= 2\kappa \sin\phi e^{-K}\left(-\frac{\dot{B}B'}{2F} - K'e^K + \dot{F}K'\right).
\end{aligned} \tag{3.15}$$

Consequently, we have

$$\Phi_g^{(2)} = -2\kappa \cos\phi \int_S dS_2 \left(\frac{\dot{B}^2}{4F^2} + \frac{B'^2}{4F^2} + K'^2\right). \tag{3.16}$$

Finally, the momentum flux

$$\Phi_g^{(3)} = -\kappa \int_S dS_1 \left(\frac{B'\dot{B}e^{-K}}{F}\right) + 2\kappa \int_S dS_3 e^{-K}\left(-\frac{B'^2}{4F} + FK' + \frac{\dot{B}^2}{4F}\right), \tag{3.17}$$

is obtained using the components of momentum flux density $\phi^{(3)j}$

$$\begin{aligned}
\phi^{(3)1} &= -\kappa\left(\frac{B'\dot{B}e^{-K}}{F}\right), \quad \phi^{(3)2} = 0, \\
\phi^{(3)3} &= 2\kappa e^{-K}\left(-\frac{B'^2}{4F} + FK' + \frac{\dot{B}^2}{4F}\right).
\end{aligned} \tag{3.18}$$

In order to evaluate matter energy-momentum flux, we have to calculate matter energy-momentum tensor. Its non-zero components are

$$\begin{aligned}
T^{00} &= \frac{e^{-2K}}{8\pi}\left(-\frac{3\dot{B}^2}{4F^2} - \frac{3B'^2}{4F^2} + \frac{\ddot{F}}{F} + \ddot{K} + K'' + \frac{7B^2\dot{B}^2}{4F^4} + \frac{7B^2B'^2}{4F^4}\right. \\
&\left. - \frac{B\dot{B}\dot{F}}{F^3} + \frac{B\ddot{B}}{F^2} + \frac{BB''}{F^2} - \frac{2B^2\ddot{F}}{F^3} - \frac{B^2\ddot{K}}{F^2} - \frac{B^2K''}{F^2}\right), \\
T^{02} &= \frac{e^{-2K}}{8\pi}\left(\frac{\dot{B}\dot{F}}{2F^3} - \frac{\ddot{B}}{2F^2} - \frac{B''}{2F^2} - \frac{B\dot{B}^2}{4F^4} - \frac{BB'^2}{4F^4} - \frac{B\ddot{K}}{F^2} - \frac{BK''}{F^2}\right) = T^{20},
\end{aligned}$$

$$
\begin{aligned}
T^{11} &= \frac{e^{-4K}}{8\pi}\left(\frac{\dot{B}^2}{4F^2}-\frac{B'^2}{4F^2}+\frac{\dot{F}\dot{K}}{F}\right),\\
T^{13} &= \frac{e^{-4K}}{8\pi}\left(\frac{B'\dot{B}}{2F^2}+\frac{K'\dot{F}}{F}\right)=T^{31},\\
T^{22} &= \frac{e^{-2K}}{8\pi}\left(\frac{\dot{B}^2}{4F^4}+\frac{B'^2}{4F^4}+\frac{\ddot{K}}{F^2}+\frac{K''}{F^2}\right),\\
T^{33} &= \frac{e^{-4K}}{8\pi}\left(-\frac{\dot{B}^2}{4F^2}+\frac{B'^2}{4F^2}-\frac{\dot{F}\dot{K}}{F}+\frac{\ddot{F}}{F}\right).
\end{aligned}
\tag{3.19}
$$

The energy flux of matter, defined in Eq.(2.11), is

$$
\Phi^{(0)}_m=\int_S dS_2 e e^{(0)}{}_\mu T^{2\mu}
$$

which gives

$$
\Phi^{(0)}_m=\frac{1}{16\pi}\int_S dS_2\left(\frac{\dot{B}\dot{F}}{F^2}-\frac{\ddot{B}}{F}-\frac{B''}{F}\right).
\tag{3.20}
$$

The components of momentum flux of matter are

$$
\begin{aligned}
\Phi^{(1)}_m &= -\frac{1}{8\pi}\sin\phi\int_S dS_2(\frac{\dot{B}^2}{4F^2}+\frac{B'^2}{4F^2}+\ddot{K}+K''),\\
\Phi^{(2)}_m &= \frac{1}{8\pi}\cos\phi\int_S dS_2(\frac{\dot{B}^2}{4F^2}+\frac{B'^2}{4F^2}+\ddot{K}+K''),\\
\Phi^{(3)}_m &= \frac{1}{8\pi}[\int_S dS_1 e^{-K}(\frac{B'\dot{B}}{2F}+K'\dot{F})+\int_S dS_3 e^{-K}(-\frac{\dot{B}^2}{4F}+\frac{B'^2}{4F}-\dot{F}\dot{K}+\ddot{F})].
\end{aligned}
\tag{3.21}
$$

Here the non-vanishing values of $\Phi^a_g$ and $\Phi^a_m$ indicate the transfer of gravitational and matter energy-momentum respectively.

# 4 Some Special Cases

In this section, we evaluate the above results for the two special cases of stationary axisymmetric Einstein-Maxwell solutions.

- **Electromagnetic Generalization of the Gödel Solution**, which is found by inserting $B=\frac{m}{n}e^{n\rho},\ F=e^{n\rho}$ and $K=0$ in Eq.(3.1).

$$ds^2=-dt^2+d\rho^2+e^{2n\rho}(1-\frac{m^2}{n^2})d\phi^2+dz^2-\frac{2m}{n}e^{n\rho}dtd\phi, \qquad (4.1)$$

- **The Gödel Metric**, obtained for $B=e^{a\rho},\ F=\frac{e^{a\rho}}{\sqrt{2}}$ and $K=0$

$$ds^2=-dt^2+d\rho^2-\frac{1}{2}e^{2a\rho}d\phi^2+dz^2-2e^{a\rho}dtd\phi, \qquad (4.2)$$

  where $m,\ n$ and $a$ are arbitrary constants with dimension of $1/L$.

The results are summarized in the following table.

**Table 1**: Energy and its contents for electromagnetic generalization of the Gödel solution and the Gödel metric

| Quantities | Electromagnetic Generalization of the Gödel Solution | The Gödel Metric |
|---|---|---|
| $P^{(0)}$ | $\frac{Le^{n\rho}}{8n}(m^2-2n^2)$ | constant |
| $P^{(1)}$ | 0 | 0 |
| $P^{(2)}$ | 0 | 0 |
| $P^{(3)}$ | constant | constant |
| $\Phi_g^{(0)}$ | constant | constant |
| $\Phi_g^{(1)}$ | $\frac{1}{2}\kappa\rho Lm^2\sin\phi$ | $\kappa La^2\rho\sin\phi$ |
| $\Phi_g^{(2)}$ | $-\frac{1}{2}\kappa\rho Lm^2\cos\phi$ | $-\kappa La^2\rho\cos\phi$ |
| $\Phi_g^{(3)}$ | $\frac{1}{16n}m^2e^{n\rho}-$constant | $\frac{1}{8\sqrt{2}}ae^{a\rho}-$constant |
| $\Phi_m^{(0)}$ | constant | constant |
| $\Phi_m^{(1)}$ | $-\frac{1}{32\pi}\rho Lm^2\sin\phi$ | $-\frac{1}{16\pi}La^2\rho\sin\phi$ |
| $\Phi_m^{(2)}$ | $\frac{1}{32\pi}\rho Lm^2\cos\phi$ | $\frac{1}{16\pi}La^2\rho\cos\phi$ |
| $\Phi_m^{(3)}$ | $\frac{1}{16n}(4n^2-m^2)e^{n\rho}+$constant | $\frac{1}{8\sqrt{2}}ae^{a\rho}+$constant |

It is interesting to mention here that energy for both cases turns out to be the same as obtained by Møller's prescription in the framework of teleparallel gravity [35]. The gravitational and matter energy fluxes are constant which depict the constant flow of energy. This indicates that the energy is getting lost at a constant rate. However, there is an outflow of gravitational

momentum along $\rho$ and $z$ direction, and inward flow along $\phi$ direction. The components of angular momentum reduce to a constant except $M^{12}$ which is $-\frac{1}{4}L\rho m$ for the electromagnetic generalization of the Gödel solution and $-\frac{1}{2\sqrt{2}}aL\rho$ for the Gödel metric. These non-zero components of angular momentum show the act of rotation in both cases.

# 5 Cosmic String Spacetimes

It is generally assumed that the universe in very early stages of its evolution has gone through a number of phase transitions. The consequence of this phase transition is the formation of topological defects, which are associated with spontaneous symmetry breaking. Cosmic strings are one of the most remarkable defects which are linear and string like. They have important implications on cosmology such as large scale structures or galaxy formation. In this section, energy contents for the non-static ($\Lambda \neq 0$) and static ($\Lambda \neq 0$) cosmic strings are evaluated.

Firstly, the energy contents for the **non-static cosmic** spacetime are discussed. The non-static line element of the cosmic string ($\Lambda \neq 0$) is [36]

$$ds^2 = -dt^2 + e^{2\sqrt{\frac{\Lambda}{3}}t}[d\rho^2 + (1-4\mu)^2\rho^2 d\phi^2 + dz^2], \tag{5.1}$$

where $\mu$ is mass per unit length of the string in geometrized units ($G = 1 = c$). The tetrad field is

$$e^a{}_\mu(t,\rho,\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A\cos\phi & -AB\rho\sin\phi & 0 \\ 0 & A\sin\phi & AB\rho\cos\phi & 0 \\ 0 & 0 & 0 & A \end{pmatrix}, \tag{5.2}$$

where $A = e^{\sqrt{\frac{\Lambda}{3}}t}$ and $B = (1-4\mu)$. Its determinant is $\rho A^3 B$. The non-zero components of $T_{\lambda\mu\nu}$ are

$$T_{101} = A\dot{A}, \quad T_{202} = A\dot{A}B^2\rho^2, \quad T_{212} = A^2B\rho(B-1), \quad T_{303} = A\dot{A}, \tag{5.3}$$

where dot denotes differentiation with respect to $t$.

The energy-momentum density components are

$$\begin{aligned} -\partial_i\Pi^{(0)i} &= 0 = -\partial_i\Pi^{(3)i}, \quad -\partial_i\Pi^{(1)i} = -16\kappa\mu\sqrt{\frac{\Lambda}{3}}e^{2\sqrt{\frac{\Lambda}{3}}t}\cos\phi, \\ -\partial_i\Pi^{(2)i} &= -16\kappa\mu\sqrt{\frac{\Lambda}{3}}e^{2\sqrt{\frac{\Lambda}{3}}t}\sin\phi \end{aligned} \tag{5.4}$$

which reflect the symmetry of the cosmic string spacetime. Thus the energy becomes constant while the momentum vanishes. Also, the angular momentum turns out to be constant.

## 5.1 Energy-Momentum Flux

The components of gravitational flux density are

$$\phi^{(0)1} = -8\kappa\mu\sqrt{\frac{\Lambda}{3}}e^{\sqrt{\frac{\Lambda}{3}}t}, \quad \phi^{(0)2} = 0 = \phi^{(0)3} \tag{5.5}$$

which in turn gives rise to the gravitational energy flux

$$\Phi_g^{(0)} = -\mu L\sqrt{\frac{\Lambda}{3}}e^{\sqrt{\frac{\Lambda}{3}}t} + \text{constant}. \tag{5.6}$$

Inserting the components of gravitational momentum flux density $\phi^{(1)i}$,

$$\phi^{(1)1} = -\frac{2\Lambda}{3}\kappa\rho\cos\phi e^{2\sqrt{\frac{\Lambda}{3}}t}(1-4\mu), \quad \phi^{(1)2} = \frac{2\Lambda}{3}\kappa\sin\phi e^{2\sqrt{\frac{\Lambda}{3}}t}, \quad \phi^{(1)3} = 0, \tag{5.7}$$

in Eq.(2.11) and integrating it over a cylindrical region of length $L$ and radius $\rho$, we obtain momentum flux $\Phi_g^{(1)}$

$$\Phi_g^{(1)} = \frac{2\Lambda}{3}\kappa\sin\phi L\rho e^{2\sqrt{\frac{\Lambda}{3}}t} + \text{constant}. \tag{5.8}$$

Similarly, the momentum flux $\Phi_g^{(2)}$,

$$\Phi_g^{(2)} = -\frac{2\Lambda}{3}\kappa\cos\phi L\rho e^{2\sqrt{\frac{\Lambda}{3}}t} + \text{constant}, \tag{5.9}$$

is found using the components of gravitational momentum flux density $\phi^{(2)i}$

$$\phi^{(2)1} = -\frac{2\Lambda}{3}\kappa\rho\sin\phi e^{2\sqrt{\frac{\Lambda}{3}}t}(1-4\mu), \quad \phi^{(2)2} = -\frac{2\Lambda}{3}\kappa\cos\phi e^{2\sqrt{\frac{\Lambda}{3}}t}, \quad \phi^{(2)3} = 0. \tag{5.10}$$

The momentum flux $\Phi_g^{(3)}$ is

$$\Phi_g^{(3)} = -\frac{\Lambda}{24}\rho^2 e^{2\sqrt{\frac{\Lambda}{3}}t}(1-4\mu) + \text{constant}, \tag{5.11}$$

obtained using $\phi^{(3)i}$

$$\phi^{(3)1}=0=\phi^{(3)2},\quad \phi^{(3)3}=-\frac{2\Lambda}{3}\kappa\rho e^{2\sqrt{\frac{\Lambda}{3}}t}(1-4\mu). \tag{5.12}$$

For matter energy-momentum flux, we evaluate the components of the matter energy-momentum tensor

$$T^{00}=\frac{\Lambda}{8\pi},\quad T^{11}=-\frac{\Lambda}{8\pi}e^{-2\sqrt{\frac{\Lambda}{3}}t}=T^{33},\quad T^{22}=-\frac{\Lambda}{8\pi}\rho^{-2}B^{-2}e^{-2\sqrt{\frac{\Lambda}{3}}t}. \tag{5.13}$$

Inserting these values in Eq.(2.11), it follows that

$$\begin{aligned}
\Phi^{(0)}_m &= \text{constant},\quad \Phi^{(1)}_m=\frac{\Lambda}{8\pi}\rho L\sin\phi e^{2\sqrt{\frac{\Lambda}{3}}t}+\text{constant},\\
\Phi^{(2)}_m &= -\frac{\Lambda}{8\pi}\rho L\cos\phi e^{2\sqrt{\frac{\Lambda}{3}}t}+\text{constant},\\
\Phi^{(3)}_m &= -\frac{\Lambda}{8}\rho^2 e^{2\sqrt{\frac{\Lambda}{3}}t}(1-4\mu)+\text{constant}.
\end{aligned} \tag{5.14}$$

Now we evaluate the energy contents for the **static cosmic string spacetime**. The general form of the static cosmic string spacetime for $\Lambda\neq 0$ $(G=1=c)$ in cylindrical polar coordinate system is [36]

$$\begin{aligned}
ds^2 &= -\cos^{\frac{4}{3}}(\frac{\sqrt{3\Lambda}}{2}\rho)dt^2+d\rho^2+\frac{4(1-4\mu)^2}{3\Lambda}\cos^{\frac{4}{3}}(\frac{\sqrt{3\Lambda}}{2}\rho)\tan^2(\frac{\sqrt{3\Lambda}}{2}\rho)d\phi^2\\
&\quad +\cos^{\frac{4}{3}}(\frac{\sqrt{3\Lambda}}{2}\rho)dz^2.
\end{aligned} \tag{5.15}$$

The corresponding tetrad field is

$$e^a{}_\mu(\rho,\phi)=\begin{pmatrix} A & 0 & 0 & 0\\ 0 & \cos\phi & -ABC\sin\phi & 0\\ 0 & \sin\phi & ABC\cos\phi & 0\\ 0 & 0 & 0 & A\end{pmatrix}, \tag{5.16}$$

where $A=\cos^{\frac{2}{3}}\alpha$, $B=\frac{2(1-4\mu)}{\sqrt{3\Lambda}}$, $C=\tan\alpha$ and $\alpha=\frac{\sqrt{3\Lambda}}{2}\rho$. The components of $T_{\lambda\mu\nu}=e^a{}_\lambda T_{a\mu\nu}$ become

$$T_{001}=A\dot{A},\quad T_{212}=ABC\{B(\dot{A}C+A\dot{C})-1\},\quad T_{313}=A\dot{A}, \tag{5.17}$$

where dot represents derivative with respect to $\rho$.

## 5.2 Energy, Momentum and Angular Momentum

The components of energy-momentum density are

$$\begin{aligned}
-\partial_i\Pi^{(0)i} &= -\partial_1[-2\kappa(A - A^2B\dot{C} - 2A\dot{A}BC)] \\
&= 2\kappa\partial_1[\cos^{\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2} + (1-4\mu)\{\frac{1}{3}\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2} \\
&\quad -\frac{4}{3}\cos^{\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\}], \\
-\partial_i\Pi^{(1)i} &= 0 = -\partial_i\Pi^{(2)i} = -\partial_i\Pi^{(3)i}. \qquad (5.18)
\end{aligned}$$

The corresponding energy becomes

$$\begin{aligned}
P^{(0)} &= \frac{L}{4}[\cos^{\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2} + (1-4\mu)\{\frac{1}{3}\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2} - \frac{4}{3}\cos^{\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\}], \\
&\approx \mu L + \frac{3L\Lambda^2\rho^2}{24} - \frac{9\mu L\Lambda^2\rho^2}{16} + \cdots \qquad (5.19)
\end{aligned}$$

and momentum turns out to be constant. Also, the components of angular momentum become constant.

## 5.3 Energy-Momentum Flux

The gravitational energy flux

$$\Phi_g^{(0)} = \text{constant} \qquad (5.20)$$

is found using the components of energy flux density $\phi^{(0)j}$

$$\phi^{(0)1} = 0 = \phi^{(0)2} = \phi^{(0)3}. \qquad (5.21)$$

The components of the gravitational momentum flux $\Phi_g^{(i)}$

$$\begin{aligned}
\Phi_g^{(1)} &= -2\kappa\frac{\Lambda}{3}\sin\phi\int_S dS_2(\sin^2\frac{\sqrt{3\Lambda}\rho}{2}\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}) + \text{constant}, \\
\Phi_g^{(2)} &= 2\kappa\frac{\Lambda}{3}\cos\phi\int_S dS_2(\sin^2\frac{\sqrt{3\Lambda}\rho}{2}\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}) + \text{constant}, \\
\Phi_g^{(3)} &= -\frac{1}{4}[\cos^{\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2} + \frac{1}{3}(1-4\mu)\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\sin^2\frac{\sqrt{3\Lambda}\rho}{2}] + \text{constant},
\end{aligned}$$
$$(5.22)$$

are obtained using the components of the gravitational momentum flux densities $\phi^{(1)j}$

$$\begin{aligned}
\phi^{(1)1} &= 2\kappa\sqrt{\frac{\Lambda}{3}}\cos\phi\sin\sqrt{3\Lambda}\rho\{(1-4\mu)-\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\},\\
\phi^{(1)2} &= -2\kappa\frac{\Lambda}{3}\sin\phi\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\sin^{2}\frac{\sqrt{3\Lambda}\rho}{2},\\
\phi^{(1)3} &= 0,
\end{aligned}\tag{5.23}$$

$\phi^{(2)j}$,

$$\begin{aligned}
\phi^{(2)1} &= 2\kappa\sqrt{\frac{\Lambda}{3}}\sin\phi\sin\sqrt{3\Lambda}\rho\{(1-4\mu)-\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\},\\
\phi^{(2)2} &= 2\kappa\frac{\Lambda}{3}\cos\phi\cos^{-\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\sin^{2}\frac{\sqrt{3\Lambda}\rho}{2},\\
\phi^{(2)3} &= 0,
\end{aligned}\tag{5.24}$$

and $\phi^{(3)j}$

$$\begin{aligned}
\phi^{(3)1} &= 0=\phi^{(3)2},\\
\phi^{(3)3} &= 2\kappa\sqrt{\frac{\Lambda}{3}}\sin\frac{\sqrt{3\Lambda}\rho}{2}[\cos^{-\frac{1}{3}}\frac{\sqrt{3\Lambda}\rho}{2}-(1-4\mu)\{\frac{1}{3}\cos^{-\frac{5}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\\
&+\frac{2}{3}\cos^{\frac{1}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\}],
\end{aligned}\tag{5.25}$$

in Eq.(2.17) for $a=(1),(2)$ and $(3)$ respectively. The non-zero components of matter energy-momentum tensor are

$$\begin{aligned}
T^{00} &= \frac{\Lambda}{8\pi}\cos^{-\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2},\quad T^{11}=-\frac{\Lambda}{8\pi},\\
T^{22} &= -\frac{3\Lambda^{2}}{32\pi}(1-4\mu)^{-2}\cos^{\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\sin^{-2}\frac{\sqrt{3\Lambda}\rho}{2},\\
T^{33} &= -\frac{\Lambda}{8\pi}\cos^{-\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2}.
\end{aligned}\tag{5.26}$$

Thus we obtain the energy flux of the matter as

$$\Phi^{(0)}_{m}=\text{constant}\tag{5.27}$$

and the components of momentum flux of matter as

$$\begin{aligned}
\Phi_m^{(1)} &= \frac{\Lambda}{8\pi}\sin\phi\int_S dS_2\cos^{\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2}+\text{constant},\\
\Phi_m^{(2)} &= -\frac{\Lambda}{8\pi}\cos\phi\int_S dS_2\cos^{\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2}+\text{constant},\\
\Phi_m^{(3)} &= \frac{1}{4}(1-4\mu)\cos^{\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2}+\text{constant}.
\end{aligned} \tag{5.28}$$

Substitution of Eqs.(5.20) and (5.27) in Eq.(2.14) provides the loss of gravitational energy

$$\frac{dE}{dt}=\text{constant}. \tag{5.29}$$

# 6 Summary and Discussion

In this paper, we have evaluated energy, momentum, angular momentum, gravitational and matter energy-momentum fluxes of stationary axisymmetric Einstein-Maxwell solutions and cosmic string spacetimes. The energy expression for stationary axisymmetric Einstein-Maxwell solutions is well-defined and coincides with the result of [35] obtained using Møller's prescription in teleparallel theory. This also shows consistency with the result of GR [37] along with some extra terms. However, for some particular values of $F$ and $K$, it reduces exactly to the energy expression evaluated in GR [37], obtained using Møller's prescription. Here the momentum components $P^{(i)}$ vanish. All the components of angular momentum except $M^{12}$ and $M^{23}$ become constant. The gravitational and matter energy-momentum fluxes are also investigated. Its non-zero components represent the transfer of gravitational and matter energy-momentum whose values depend on the metric coefficients $B,\ F$ and $K$. We have also evaluated the above quantities for the two special cases of stationary axisymmetric Einstein-Maxwell solutions (electromagnetic generalization of the Gödel solution and Gödel metric), obtained for specific values of the metric functions $B,\ F$ and $K$.

This procedure has been extended to non-static and static cosmic strings, for $\Lambda \neq 0$. For the static cosmic string, energy is finite and well-defined whereas momentum turns out to be constant. For the non-static cosmic string, we have constant energy. If we take this constant to be zero then the energy coincides with the result of GR [14]. However, the angular momentum

for both the cases become constant. When $\Lambda \rightarrow 0$, both the metrics (5.1) and (5.15) reduce to the static cosmic string ($\Lambda = 0$). In this case, the energy becomes constant and momentum vanishes. When we choose this constant to be zero, the result agrees with that of GR [14]. The components of $\Phi_g^a$ and $\Phi_m^a$ are well-defined for both cosmic string spacetimes. The flow of energy is towards the source for non-static cosmic string whereas the momentum flows outward along radial direction and inwards along $\phi$ and $z$ direction. For the static cosmic string there is a constant flow of energy. Thus we can say that the loss of gravitational energy is constant. For $\Lambda \rightarrow 0$, the gravitational energy-momentum flux becomes constant. However, matter energy-momentum flux vanishes because the static cosmic string ($\Lambda = 0$) is a vacuum solution of the EFEs. If we set $\Lambda = 0 = \mu$ in metrics (5.1) and (5.15), then the cosmic string spacetimes reduce to the Minkowski spacetime and energy-momentum becomes zero as expected due to Minkowski spacetime.

In the perspective of the above discussion, we can conclude that the gravitational energy-momentum revealed by the prescription [24] shows the correspondence with the results of different energy-momentum complexes, particularly with Møller's prescription, both in GR and teleparallel gravity. It is interesting to mention here that this Hamiltonian approach to define the conserved quantities yield consistent results in most of the cases with those already available in the literature. It would be worthwhile to investigate this problem deeply to get more comprehension about it which may lead to some indication about its well-defined and unique solution.

# Appendix

The non-zero components of the tensor $\Sigma^{abc}$ for

- **Stationary Axisymmetric Einstein-Maxwell Solutions**

$$\begin{aligned}
\Sigma^{001} &= \frac{e^{-K}}{2}\left(\frac{B\dot{B}}{F^2}e^{-K} + \frac{B^2\dot{K}}{F^2}e^{-K} + \frac{1}{F} - \frac{\dot{F}}{F}e^{-K} - \dot{K}e^{-K}\right), \\
\Sigma^{003} &= \frac{e^{-2K}}{2}\left(\frac{BB'}{F^2} - \frac{B^2K'}{F^2} + K'\right), \\
\Sigma^{012} &= \frac{e^{-2K}}{2F^2}\left(\frac{\dot{B}}{2} + B\dot{K}\right) = -\Sigma^{201}, \\
\Sigma^{023} &= \frac{e^{-2K}}{2F^2}\left(BK' - \frac{B'}{2}\right) = \Sigma^{203}, \\
\Sigma^{102} &= \frac{\dot{B}}{4F^2}e^{-2K}, \quad \Sigma^{113} = -K'e^{-4K}, \\
\Sigma^{212} &= -\frac{\dot{K}}{2F^2}e^{-2K}, \quad \Sigma^{223} = -\frac{K'}{2F^2}e^{-2K}, \\
\Sigma^{302} &= \frac{B'}{4F^2}e^{-2K}, \quad \Sigma^{313} = \frac{1}{2F}e^{-3K}(1 - \dot{F}e^{-K}),
\end{aligned}$$

- **Non-Static Cosmic String**

$$\begin{aligned}
\Sigma^{001} &= 2\mu\rho^{-1}(1-4\mu)^{-1}e^{-2\sqrt{\frac{\Lambda}{3}}t}, \\
\Sigma^{101} &= \sqrt{\frac{\Lambda}{3}}e^{-2\sqrt{\frac{\Lambda}{3}}t} = \Sigma^{303}, \\
\Sigma^{202} &= \sqrt{\frac{\Lambda}{3}}\rho^{-2}(1-4\mu)^{-2}e^{-2\sqrt{\frac{\Lambda}{3}}t}, \\
\Sigma^{313} &= 2\mu\rho^{-1}(1-4\mu)^{-1}e^{-4\sqrt{\frac{\Lambda}{3}}t},
\end{aligned}$$

- **Static Cosmic String**

$$\begin{aligned}
\Sigma^{001} &= \frac{1}{2\sin\sqrt{3\Lambda}}\sqrt{\frac{\Lambda}{3}}(1-4\mu)^{-1}\{3 + (1-4\mu)(\cos^{-\frac{4}{3}}\frac{\sqrt{3\Lambda}\rho}{2} \\
&\quad -4\cos^{\frac{2}{3}}\frac{\sqrt{3\Lambda}\rho}{2})\} = \Sigma^{313}, \\
\Sigma^{212} &= \frac{\sqrt{3}}{4}\Lambda^{\frac{3}{2}}(1-4\mu)^{-2}\cos^{-\frac{1}{3}}\frac{\sqrt{3\Lambda}\rho}{2}\sin^{-1}\frac{\sqrt{3\Lambda}\rho}{2}.
\end{aligned}$$

## References

[1] Møller, C.: *The Theory of Relativity* (Oxford University Press, Oxford, 1957).

[2] Møller, C.: Ann. Phys. **4**(1958)347.

[3] Landau, L.D. and Lifshitz, E.M.: *The Classical Theory of Fields* (Addison-Wesley Press, New York, 1962).

[4] Papapetrou, A.: *Proc. R. Irish Acad.* **A52**(1948)11.

[5] Bergmann, P.G. and Thompson, R.: Phys. Rev. **D89**(1953)400.

[6] Tolman, R.C.: *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Oxford, 1934).

[7] Weinberg, S.: *Gravitation and Cosmology: Principle and Applications of General Theory of Relativity* (Wiley, New York, 1972).

[8] Komar, A.: Phys. Rev. **113**(1959)934.

[9] Virbhadra, K.S.: Phys. Rev. **D60**(1999)104041.

[10] Virbhadra, K.S. and Parikh, J.C.: Phys. Lett. **B331**(1994)302.

[11] Radinschi, I.: Acta Phys. Solv. **50**(2000)609.

[12] Banerjee, N. and Sen, S.: Pramana **49**(1997)609.

[13] Xulu, S.S.: Int. J. Theor. Phys. **39**(2000)1153.

[14] Abbassi, A.M., Mirshekari, S. and Abbassi, A.H.: Phys. Rev. **D78**(2008)064053.

[15] Sharif, M.: Int. J. Mod. Phys. **A17**(2002)1175; Int. J. Mod. Phys. **D13**(2004)1019; Braz. J. Phys. **37**(2007)1292.

[16] Nashed, G.G.L.: Phys. Rev. **D66**(2002)060415.

[17] Xu, S. and Jing, J.: Class. Quantum Grav. **23**(2006)4659.

[18] Møller, C.: *Tetrad Fields and Conservation Laws in General Relativity* (Academic Press, London, 1962).

[19] Mikhail, F.I., Wanas, M.I., Hindawi, A. and Lashin, E.I.: Int. J. Theo. Phys. **32**(1993)1627.

[20] Loi So, L. and Vargas, T.: Chinese J. Phys. **43**(2005)901.

[21] Sharif, M. and Amir, M. Jamil.: Mod. Phys. Lett. **A22** (2007)425.

[22] Sharif, M. and Amir, M. Jamil.: Int. J. Theor. Phys. **47**(2008)1742.

[23] de Andrade, V.C., Guillen, L.C.T. and Pereira, J.G.: Phys. Rev. Lett. **84**(2000)4533.

[24] Maluf, J.W., da Rocha-Neto, J.F., Toribio, T.M.L. and Castello-Branco, K.H.: Phys. Rev. **D65**(2002)124001.

[25] Maluf, J.W.: J. Math. Phys. **35**(1994)335.

[26] Maluf, J.W.: J. Math. Phys. **37**(1996)6293.

[27] Maluf, J.W.: Annalen Phys. **14**(2005)723.

[28] da Rocha-Neto, J.F. and Castello-Branco, K.H.: JHEP **11**(2003)002.

[29] Maluf, J.W. and Ulhoa, S. C.: Phys. Rev. **D78**(2008)069901(E).

[30] Sharif, M. and Taj, S.: Mod. Phys. Lett. **A**(2009, to appear).

[31] Aldrovandi, R. and Pereira, J.G.: *An Introduction to Gravitation Theory* (preprint).

[32] Maluf, J.W. and da Rocha-Neto, J.F.: Phys. Rev. **D64**(2001)084014.

[33] Maluf, J.W. and Faria, F.F.: Annalen Phys. **13**(2004)604.

[34] Tupper, B.O.J.: Class. Quantum Garv. **2**(1985)427.

[35] Sharif, M. and Amir, M. Jamil.: Gen. Relativ. Gravit. **39**(2007)989.

[36] Abbassi, A.H., Abbassi, A.M. and Razmi, H.: Phys. Rev. **D67**(2003)103504.

[37] Sharif, M. and Fatima, T.: Int. J. Mod. Phys. **A20**(2005)4309.